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On Probability Functions based on Assumptions of the Relative Likelihood among Events

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Abstract
In probability and statistics much works deals with how to assign probability to a certain events with the study of the manner in which certain specified probabilities change in the light of new information. In this paper we shall discuss the assumptions under which we can represent these information and beliefs in terms of probability distributions, so we shall consider the following question "how does we initially assign to these events the probabilities based on so-called relative likelihoods among events.

1. INTRODUCTION

In many problems the assignment of a probability distribution over the relevant σ-filed $\mathcal{F}$ of events become routine and standardized among workers in those area, in some other problems there are natural probability models that represent the situations. Hence we interpret probability as a measure of individual's personal belief in a particular event and suitable probabilities can often be assigned objectively and quickly because of wide agreement of a specific distribution for a certain type of problem.

2. Relative Likelihood Suppose we have a sample space S together with σ-filed $\mathcal{F}$ and suppose that its desired to assign a probability to each event in σ-filed $\mathcal{F}$ and assumed that 2.1 A person when considering any two events can decide whether one of them as being more likely to occurs than the other or the two events being equally likely to occurs, so we have the symbols

$A \prec B$ to indicate that B more likely to occurs than A.

$A \sim B$ to indicate that A and B are equally likely to occur.
$A \precsim B$ to indicate that B is at least as likely to occurs as A or A is not more likely to occur than B.

Hence if $A \precsim B$, then either $A \prec B$ or $A \sim B$.

Any probability distribution which assigned to the events in the σ-filed $\mathcal{F}$ have the property $P(A) \leq P(B)$ if and only if $A \precsim B$, a probability which has this property is said to agree with the relation $\precsim$.

We will introduce to the basic assumptions.

2.1 Assumption 1 For any events A and B, one of the following relations must hold: $A \prec B$, $A \succ B$, $A \sim B$.

2.2 Assumption 2 If A_1, A_2, B_1, and B_2 are four events such that $A_1A_2 = B_1B_2 = \emptyset$, and $A_i \precsim B_i$, for *i = 1, 2*, then $A_1 \cup A_2 \precsim B_1 \cup B_2$, and if in addition either $A_1 \prec B_1$ or $A_2 \prec B_2$ then $A_1 \cup A_2 \prec B_1 \cup B_2$.

2.3 Assumption 3 If A is any events , then $\emptyset \precsim$ A, as well as $\emptyset \prec$ A.

2.4 Assumption 4 If $A_1 \supset A_2 \supset$ .... is a decreasing sequence of events and if B is some fixed event such that $A_i \succsim B$, for *i = 1, 2*,....then $\bigcap_{i=1}^{\infty} A_i \succsim B$.

Several results can be derived from the first two assumptions, one interesting and im-portant result is the transivity of the relation $\precsim$.

Before proving this theorem we shall present a simple lemma.

2.5 Lemma Suppose that A , B ,and D are three events such that AD = BD = $\emptyset$, then A $\precsim$ B , if and only if A ∪ D $\precsim$ B ∪ D.

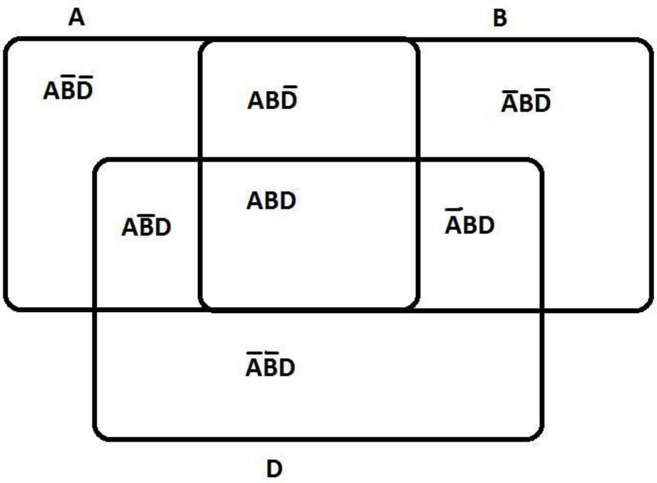
Proof

Proof of this lemma follows from assumption 2 by substituting

A1 = A, B1 = B and A2 = B2 = D that is A $\precsim$ B $\Leftrightarrow$ A1 ∪ A2 $\prec$ A∪ D $\precsim$ B1 ∪ B2 = B ∪ D, where A1 ∩ A2 $\prec$ A ∩ D $\precsim$ B1 ∩ B2 = B ∩ D = $\emptyset$. Consequently if A $\succ$ B, then again by assumption 2 that A ∪ D $\succ$ B ∪ D.

2.6 Theorem If A, B ,and D are three events such that A $\precsim$ B and B $\precsim$ D, then A $\precsim$ D.

Proof



Consider the seven disjoint events shown in figure 2.6 whose union is A ∪ B ∪ D it's known that

$$A = (A - B) \cup (A \cap B) = A\bar{B}\bar{D} \cup A\bar{B}D \cup AB\bar{D} \cup ABD$$

$$B = (B - A) \cup (A \cap B) = \bar{A}B\bar{D} \cup \bar{A}BD \cup AB\bar{D} \cup ABD$$

from lemma 2.5 it follows that

$$A \precsim B \Leftrightarrow [(A-B) \cup (A \cap B) \precsim (B-A) \cup (A \cap B)] \Leftrightarrow [A-B \precsim B-A]$$

$$\Leftrightarrow A\bar{B}\bar{D} \cup A\bar{B}D \precsim \bar{A}B\bar{D} \cup \bar{A}BD \qquad (2.1)$$

, similarly since $A \precsim B$, it follows again from Lemma 2.5 that

$$[(B-D) \cup (B \cap D) \precsim (D-B) \cup (B \cap D)] \Rightarrow [B-D \precsim D-B]$$

$$\Leftrightarrow \bar{A}B\bar{D} \cup AB\bar{D} \precsim A\bar{B}D \cup \bar{A}\bar{B}D \qquad (2.2)$$

Since the left and the right sides of relation 2.1, 2.2 are disjoint. From assumption 2.1 it follows that

$$A\bar{B}\bar{D} \cup A\bar{B}D \cup AB\bar{D} \cup \bar{A}B\bar{D} \precsim \bar{A}B\bar{D} \cup$$

$$\bar{A}BD \cup A\bar{B}D \cup \bar{A}\bar{B}D \qquad (2.3)$$

By eliminate the term $A\bar{B}D \cup \bar{A}B\bar{D}$ from the both sides of (2.3) we get

$$A\bar{B}\bar{D} \cup AB\bar{D} \precsim \bar{A}BD \cup \bar{A}\bar{B}D$$

Now it can be seen from Fig 2.6 and lemma 2.5 that $A \precsim D$.

2.7 Corollary If A, B ,and D are any three events such that $A \precsim B$ B and $B \prec D$, then $A \prec D$.

Proof

Again from the figure 2.6 and by the same way

$$A \precsim B \Rightarrow A\bar{B}\bar{D} \cup A\bar{B}D \precsim \bar{A}B\bar{D} \cup \bar{A}BD \qquad (2.4)$$

$$B = (B - D) \cup (B \cap D)$$

$$D = (D - B) \cup (B \cap D)$$

Since it follows from lemma 2.5 that

$$B \prec D \Leftrightarrow [(B - D) \cup (B \cap D) \prec (D - B) \cup (B \cap D)] \Leftrightarrow [B - D \prec D - B]$$

$$\Leftrightarrow AB\bar{D} \cup \bar{A}B\bar{D} \prec A\bar{B}D$$

$$\cup \bar{A}\bar{B}D \qquad (2.5)$$

Since the left and the right sides of relation 2.4, 2.5 are disjoint again from assumption 2.1

We have

.

$$A\bar{B}\bar{D} \cup A\bar{B}D \cup AB\bar{D} \cup \bar{A}B\bar{D} \prec \bar{A}B\bar{D} \cup \bar{A}BD \cup A\bar{B}D \cup \bar{A}\bar{B}D \Rightarrow A\bar{B}\bar{D} \cup AB\bar{D} \prec \bar{A}BD \cup \bar{A}\bar{B}D \Rightarrow A \prec D$$

2.8 Theorem

If A, B are any two events then A $\lesssim$ B if and only if $\bar{A} \gtrsim \bar{B}$.

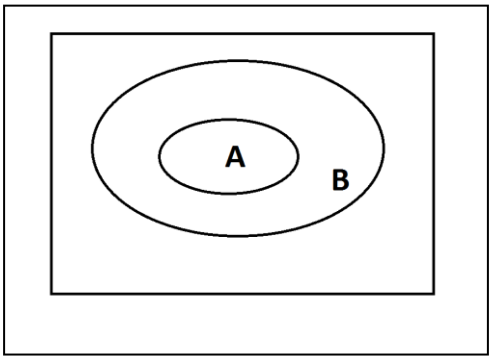
Proof

From lemma 2.5 we have

$$A \lesssim B \Leftrightarrow [(A\text{-}B) \cup (A \cap B) \lesssim (B\text{-}A) \cup (A \cap B)] \Leftrightarrow [A\text{-}B \lesssim B\text{-}A] \Leftrightarrow [B - A \gtrsim A\text{-}B] \Leftrightarrow [\bar{A} \cap B \gtrsim A \cap \bar{B}] \Leftrightarrow [\bar{A} \cap \bar{\bar{B}} \gtrsim \bar{\bar{A}} \cap \bar{B}] \Leftrightarrow \bar{A}\text{-}\bar{B} \gtrsim \bar{B}\text{-}\bar{A} \Leftrightarrow [(\bar{A}\text{-}\bar{B}) \cup (\bar{A} \cap \bar{B}) \gtrsim (\bar{B} - \bar{A}) \cup (\bar{A} \cup \bar{B})] \Leftrightarrow \bar{A} \gtrsim \bar{B}.$$

2.9 Theorem If A, B are any two events, such that A ⊂ B then A $\lesssim$B.

Proof



A-B=∅

$\emptyset$ = A-B $\lesssim$ B-A $\Rightarrow$ A = (A-B) ∪ (A ∩ B) $\lesssim$ B = (B-A) ∩ (A ∩ B)

If A ⊂ B, then It follows from assumption 2.3 that ∅ = A-B ⊃ B-A. From lemma 2.5 we have
(A-B) ∪ (A ∩ B) $\lesssim$ (B - A) ∩ (A ∩ B), because

(A-B) ∪ (A ∩ B) = A and (B - A) ∩ (A ∩ B) = B, then A $\precsim$B.

2.10 Theorem For any event A , A $\precsim$ *S*.

Proof

From assumption 2.3 and theorem 2.9 we have

$$\bar{\emptyset} \precsim \bar{A} \Leftrightarrow \bar{\emptyset} \succsim \bar{\bar{A}} \Leftrightarrow S \succsim A.$$

The following theorem is the dual of assumption 2.4 and it could be taken as assumption.

2.11 Theorem

If $A_1 \subset A_2 \subset \ldots$ is an increasing sequence of events and if B is some fixed event such that $A_i \precsim B$ for *i* = *1, 2*,.... then $\bigcup_{i=1}^{\infty} A_i \precsim B$.

Proof

If $A_1 \subset A_2 \subset \ldots$ is an increasing sequence of events and if B is some fixed event such that $\bar{A}_i \succsim \bar{B}_i$ for *i* = *1, 2*,.... hence by assumption 2.4 we have
$\bigcap_{i=1}^{\infty} \bar{A}_i \succsim \bar{B} \Leftrightarrow \overline{\left(\bigcup_{i=1}^{\infty} A_i\right)} \succsim \bar{B} \Leftrightarrow (again\ by\ theorem\ 2.8)\left(\bigcup_{i=1}^{\infty} A_i\right) \precsim B.$

2.12 Theorem

If A1 , A2,… is an infinite sequence of disjoint events and if B1,B2, …
is another infnite sequence of disjoint events such that. $A_i \precsim B_i$ for *i = 1, 2*,….,then $\bigcup_{i=1}^{\infty} A_i \precsim \bigcup_{i=1}^{\infty} B_i$. If in addition $A_i \prec B_i$ for at least one value of *i* , *i= 1, 2*,…., then $\bigcup_{i=1}^{\infty} A_i \prec \bigcup_{i=1}^{\infty} B_i$.

Proof

it follows from assumption 2.2 that $\bigcup_{i=1}^{n} A_i \precsim \bigcup_{i=1}^{n} B_i$ for any value of *n* , *n= 1, 2*,….,, hence by theorem 2.9 we have

$$\bigcup_{i=1}^{n} A_i \precsim \bigcup_{i=1}^{\infty} B_i\ , n = 1,2,\ldots \qquad (2.6)$$

it follows from theorem 2.13 (because the left side of relation 2.6 yields an increasing sequence of events, $B = \bigcup_{i=1}^{\infty} B_i$ that $\bigcup_{i=1}^{\infty} A_i \precsim \bigcup_{i=1}^{\infty} B_i$.

If in addition Ai ≺ Bi for at least one value of of i , i= 1, 2,…., then It follows again from assumption 2.2 that

$$\bigcup_{i=1}^{n} A_i \prec \bigcup_{i=1}^{n} B_i \qquad (2.7)$$

Furthermore, it follows from the first part of theorem 2.14 which has been just proved that

$$\bigcup_{i=n+1}^{\infty} A_i \precsim \bigcup_{i=n+1}^{\infty} B_i \qquad (2.8)$$

From the relations 2.7, 2.8 and assumption 2.2 we can obtain the following result

$$\bigcup_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=n+1}^{\infty} A_i\right) \prec \left(\bigcup_{i=1}^{n} B_i\right) \cup \left(\bigcup_{i=n+1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} B_i$$

In the end of this section we shall present some results (without proofs) which are follows from assumptions 2.1 to 2.4.

2.13 Theorem

If $A1 \supset A2 \supset \ldots$ is a decreasing sequence of events and if B is an event such that $\bigcap_{i=1}^{\infty} A_i \prec B$, then $A_i \gtrsim B$, only for a finite number of i.

2.14 Theorem

If $A1 \subset A2 \subset \ldots$ is an increasing sequence of events and $B1 \supset B2 \supset \ldots$ is a decreasing sequence of events such that $A_i \precsim B_i$ for i = 1, 2,…. then $\bigcup_{i=1}^{\infty} A_i \precsim \bigcap_{i=1}^{\infty} B_i$.

2.15 Theorem

If $A1 \supset A2 \supset \ldots$ is a decreasing sequence of events and $B1 \supset B2 \supset \ldots$ is also decreasing sequence of events such that $A_i \precsim B_i$ for i = 1, 2,…. then $\bigcap_{i=1}^{\infty} A_i \precsim \bigcap_{i=1}^{\infty} B_i$. Furthermore if
$A_i \sim B_i$ for i = 1, 2,…. then $\bigcup_{i=1}^{\infty} A_i \sim \bigcap_{i=1}^{\infty} B_i$, and $B1 \subset B2 \subset \ldots$ is an increasing sequence of events such that $A_i \precsim B_i$ for i = 1, 2,…. then $\bigcup_{i=1}^{\infty} A_i \precsim \bigcup_{i=1}^{\infty} B_i$.

Now we will assigning a numerical probability to each event of those comparable relative likelihoods on the bases of the assumptions which have been made before and agree with these relations likelihoods.

2.16 Theorem If A, B are any two events then A $\lesssim$ B if and only if $P(A) \leq P(B)$.

Proof

if $P(A) \leq P(B)$ holds $\Leftrightarrow$ $1 - P(A) \geq 1 - P(B) \Leftrightarrow P(\bar{A}) \geq P(\bar{B})$ $\Leftrightarrow$ $\bar{A} \gtrsim \bar{B}$ $\xLeftrightarrow{\text{by theorem } 2.6}$ A $\lesssim$ B.

This is true and agree with any relation likelihoods. Also we can state;

If the relation likelihood satisfies all the previous assumptions then the function P is the probability distribution which agrees with theses relations.

3 Conditional Likelihood In this section we shall extend the concept of relation A $\lesssim$ B to the conditional likelihood (A | D) $\lesssim$ (B | D) for any three events A, B, and D. So according to theorem 2.16, holds if and only if P(A | D) $\leq$ P(B | D). Clearly this equivalent can be required only for the event D, such that P(D) > 0, since conditional probability is defined only for such event. For any event D, such that P (D) > 0, the inequality P(A | D) $\leq$ P(B | D) ~ P(AD) $\leq$ P(BD). Furthermore, this inequality is equivalent to the relation AD $\lesssim$ BD.

With these remarks naturally leads to the next assumption.

3.1 Assumption 5 For any three events A, B, and D, (A | D) $\lesssim$ (B | D), if and only if AD $\lesssim$ BD.

As we remarked above, when Assumption 2.1 to 2.4 are made and there exists a probability distribution P, which can be uniquely specified, its sufficient for most purposes to apply Assumption 3.1 only to events D such that P(D) > 0. Furthermore, it now follows from Assumptions 2.1 and 3.1 that for any three events A, B, and D either (A | D) $\lesssim$ (B | D) or (B | D) $\lesssim$ (A | D), both relation will always be correct when is an event such that P (D) = 0.

Finally we will present some Theorems without proofs which are results follow from the Assumptions 2.1 to 3.1.

3.2 Theorem If A, B are any two events, such that A $\subset$ B then (A | D) $\lesssim$ (B | D) for any event D.

3.3 Theorem For any three events A, B, and D, then (A | D) $\lesssim$ (B | D), if and only if ($\bar{A}$ | D) $\gtrsim$ ($\bar{B}$ | D) .

3.4 Theorem If A1 ⊃ A2⊃… is a decreasing sequence of events and if B, D are any given events such that $(A_i \mid D) \gtrsim (B \mid D)$ for $i = 1, 2, \ldots,$ then $(\bigcap_{i=1}^{\infty} A_i \mid D) \gtrsim (B \mid D)$.

3.5 Theorem If D1,D2, … be events such that $D_i D_j \sim \emptyset$ for all $i \neq j$ and $\bigcap_{i=1}^{\infty} D_i \sim S$. And if A, B are any given events such that $(A \mid D_i) \gtrsim (B \mid D_i)$ for $i = 1, 2, \ldots,$ then $A \lesssim B$. If in addition $(A \mid D_i) \prec (B \mid D_i)$ for at least one value of $i = 1, 2, \ldots,$ then $A \prec B$.

4. CONCLUSION

Since the relations likelihood satisfies all the given assumptions in this article, hence the function P is the probability that agrees with theses relations. Consequently, and as we have seen most of the probability objective including conditional probabilities with help of these assumptions can be proved in terms of the comparable likelihoods relations.

5. REFERENCES

[1] Berner, K.R.W (1963). Decision Under Uncertainty.

[2] Christian P. Robert (1961). The Bayesian Choice A Desicion- Theoretic Motivation.

[3] DeGroot, M.H. (1970). Optimal Statitical Deciosions, New York.

[4] Fellner, W (1961). Distortion of Subjective Probability As a Reaction to Uncertainty, Quart. J. Econ.

[5] Ik-Whan-Kwon, Statistical Decision Theory With Bussiness And Economic Applications, New York- 1978.

[6] Von Neuman, I., and Morgenstern, O. (1947). Theory Of Games And Economic Behavior, Zd ed. Princeton University Press, Princeton, N.J.

[7] Raia, and Schlaifer, R. (1961). Applied Statistical Decision Theory, Division Of Research, Gradnate School Of Businees Administration, Havard University, Boston.
Pratt, I.W (1964). Risk Aversion In The Small And In The Large.